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Communication

On a problem of P. Erdős

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Abstract

A minimum problem proposed by P. Erdős is considered and results about the existence and uniqueness of solutions are given.

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1. Introduction

Recently P. Erdős [1] has proposed the following problem: "for an arbitrary triangle T (understood as a convex set of the Euclidean plane) and arbitrary point X, let a, b, c and x, y, z denote distances of X from vertices and sides of T, respectively. When does the ratio

$$\frac{a+b+c}{x+y+z}$$

have the minimum value?" Obviously the Erdős problem collapses to the classic Fermat problem [3, 6] when $x+y+z$ is constant (for example, when T is equilateral and X must belong to T).

Without any loss of generality, we fix in the Euclidean plane a reference system with the origin at one of the vertices of T. Moreover, let X_1, X_2 be the vectors of the plane identified by the other two points, and let $\widehat{X}_1$, $\widehat{X}_2$, $\widehat{X}_3$ be respectively the vectors $X_1/\|X_1\|$, $X_2/\|X_2\|$ and $(X_1-X_2)/\|X_1-X_2\|$ after a counterclockwise rotation of $\pi/2$ radians has been performed. Then, the problem can be formulated in the following way:

$$\min_{X\in\mathbb{R}^2} F(X),$$

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where $F : \mathbb{R}^2 \to]1, +\infty[$, with

$$F(X) := \frac{\| X \| + \| X - X_1 \| + \| X - X_2 \|}{|\langle X, \widehat{X}_1 \rangle| + |\langle X, \widehat{X}_2 \rangle| + |\langle X - X_1, \widehat{X}_3 \rangle|} := \frac{N(X)}{D(X)},$$

and where $\| \cdot \|$ and $\langle \cdot, \cdot \rangle$ denote the Euclidean norm and the usual scalar product, respectively.

2. Existence and uniqueness of solutions

The following results concern the existence and the uniqueness of the solutions. Without any loss of generality, assume the vectors X_1 and X_2 such that

$$\langle X_1, X_2 \rangle > 0, \quad \| X_2 \| \geqslant \| X_1 - X_2 \| \geqslant \| X_1 \| .$$

Moreover, for any non-null vector X in the plane, $\widetilde{X}$ denotes the vector $X / \| X \|$. Instead of F we will study the function $G := 1/F$. To this end consider $S := \{X \in \mathbb{R}^2 : \| X \| = 1\}$ and the function $G_\infty : S \to]0, 1[$ defined by

$$G_\infty(W) := \lim_{\substack{t \in \mathbb{R} \\ |t| \to \infty}} G(tW) = \tfrac{1}{3}(|\langle W, \widehat{X}_1 \rangle| + |\langle W, \widehat{X}_2 \rangle| + |\langle W, \widehat{X}_3 \rangle|).$$

Lemma 1. *Let W_η denote the vector $\widehat{X}_1 + \alpha \widehat{X}_2 + \beta \widehat{X}_3$, with $\eta := (\alpha, \beta) \in \{-1, 1\}^2$, and set $\| W_* \| := \max_\eta \| W_\eta \|$. The following relations hold*:

$$\max_W G_\infty(W) = G_\infty(\widetilde{W}_*),$$

$$|G(tW) - G_\infty(W)| \leqslant 2(\| X_1 \| + \| X_2 \|)|t|^{-1}.$$

Proof. As concerns the former relation, observe that

$$\tfrac{1}{3} \| W_\eta \| = \tfrac{1}{3} |\langle \widetilde{W}_\eta, W_\eta \rangle| \leqslant G_\infty(\widetilde{W}_\eta) \leqslant \sup_W G_\infty(W) \leqslant \tfrac{1}{3} \sup_\eta \| W_\eta \|,$$

and hence we have

$$\sup_W G_\infty(W) = \tfrac{1}{3} \sup_\eta \| W_\eta \| .$$

The latter relation holds obviously. □

Theorem 1. *The function F has the minimum.*

Proof. Because of the second relation in the lemma, the existence of a number t_*, such that

$$G_\infty(\widetilde{W}_*) < G(t_* \widetilde{W}_*),$$

proves that the function G has the maximum. More precisely, we prove that the function $g : \mathbb{R} \to \mathbb{R}$, with

$$g(v) := \frac{|\langle \widetilde{W}_*, \widehat{X}_1 \rangle| + |\langle \widetilde{W}_*, \widehat{X}_2 \rangle| + |\langle \widetilde{W}_* - vX_1, \widehat{X}_3 \rangle|}{1 + \| \widetilde{W}_* - vX_1 \| + \| \widetilde{W}_* - vX_2 \|},$$

has non-null derivative at $v = 0$. Indeed, since $\langle \widehat{X}_3, \widetilde{W}_{\eta_+} \rangle < 0$ and $\langle \widehat{X}_3, \widetilde{W}_{\eta_-} \rangle > 0$, $g'(0)$ exists and

$$g'(0) = \tfrac{1}{3}(-\beta \langle \widehat{X}_3, X_1 \rangle + G_\infty(W_*) \langle \widetilde{W}_*, X_1 + X_2 \rangle).$$

Since $G_\infty(\widetilde{W}_*) = \frac{1}{3} \| W_* \|$, $g'(0) = 0$ would imply

$$\langle W_*, X_1 + X_2 \rangle = 3\beta \langle \widehat{X}_3, X_1 \rangle.$$

Hence, by subtracting from both sides the quantity $2\beta \langle \widehat{X}_3, X_1 \rangle$ we would obtain the equalities

$$(\alpha \| X_1 \| - \| X_2 \|) \langle \widetilde{X}_1, \widehat{X}_2 \rangle = \beta \langle \widehat{X}_3, X_1 \rangle = -\beta \frac{\| X_1 \| \| X_2 \|}{\| X_1 - X_2 \|} \langle \widetilde{X}_1, \widehat{X}_2 \rangle.$$

The last is false, since the inequalities

$$\| \widehat{X}_1 + \widehat{X}_2 + \widehat{X}_3 \| \leqslant \| \widehat{X}_1 + \widehat{X}_2 - \widehat{X}_3 \|,$$

$$\| \widehat{X}_1 - \widehat{X}_2 - \widehat{X}_3 \| \leqslant \| \widehat{X}_1 - \widehat{X}_2 + \widehat{X}_3 \|$$

imply that the vector W_* is either W_{η_-} or W_{η_+} where $\eta_- = (1, -1)$ and $\eta_+ = (-1, 1)$. Therefore the maximum of G exists. □

With regard to the uniqueness, denote by C_η^+ and C_η^- the following convex sets,

$$\{X \in \mathbb{R}^2 \colon \langle X, \widehat{X}_1 \rangle = \alpha |\langle X, \widehat{X}_1 \rangle|,\ \langle X, \widehat{X}_2 \rangle = \beta |\langle X, \widehat{X}_2 \rangle|,\ \langle \widehat{X}_3, X - X_1 \rangle \geqslant 0\}$$

and

$$\{X \in \mathbb{R}^2 \colon \langle X, \widehat{X}_1 \rangle = \alpha |\langle X, \widehat{X}_1 \rangle|, \langle X, \widehat{X}_2 \rangle = \beta |\langle X, \widehat{X}_2 \rangle|,\ \langle \widehat{X}_3, X - X_1 \rangle < 0\},$$

respectively. The following result holds.

Theorem 2. *The function F has not more than one minimum point in each set C_η^+, C_η^-.*

Proof. We find

$$G(X) = \begin{cases} \dfrac{\langle X, Y_\eta^+ \rangle - \langle \widehat{X}_3, X_1 \rangle}{\| X \| + \| X - X_1 \| + \| X - X_2 \|} & \text{if } X \in C_\eta^+, \\[2ex] \dfrac{\langle X, Y_\eta^- \rangle - \langle \widehat{X}_3, X_1 \rangle}{\| X \| + \| X - X_1 \| + \| X - X_2 \|} & \text{if } X \in C_\eta^-, \end{cases}$$

where

$$Y_\eta^+ = \alpha\widehat{X}_1 + \beta\widehat{X}_2 + \widehat{X}_3, \quad Y_\eta^- = \alpha\widehat{X}_1 + \beta\widehat{X}_2 - \widehat{X}_3.$$

Ab absurdo, let V_1, V_2 be two maximum points for G in C_η^+ (C_η^- respectively). Put

$$\bar{V} = \tfrac{1}{2}(V_1 + V_2), \quad \mathcal{M} = \max G,$$

$$N(X) = \| X \| + \| X - X_1 \| + \| X - X_2 \|.$$

Now, the convexity of the function N leads to the following relations,

$$\begin{aligned} N(\bar{V})G(\bar{V}) &= \langle \bar{V}, Y_\eta^+\rangle - \langle \widehat{X}_3, X_1\rangle \\ &= \mathcal{M}\big(\tfrac{1}{2}(N(V_1) + N(V_2))\big) > \mathcal{M}N(\bar{V}), \\ (N(\bar{V})G(\bar{V}) &= \langle \bar{V}, Y_\eta^-\rangle + \langle \widehat{X}_3, X_1\rangle \\ &= \mathcal{M}\big(\tfrac{1}{2}(N(V_1) + N(V_2))\big) > \mathcal{M}N(\bar{V})), \end{aligned}$$

and hence to a contradiction. □

3. Further developments

The problem posed by Erdős leads to several questions, which presently are open. Some of them require new investigations on the Fermat problem and its elegant resolution conceived by Torricelli [3, 4]; in spite of their age, it seems that they have something more to give to Mathematics.

When T is equilateral, it is easy to show that F has 3 (global) minimum points; when T is not equilateral the uniqueness of the solution is an open question.

The Euclidean plane can be partitioned into lines on which the denominator of F is constant. On each line the Erdős problem becomes a "constrained Fermat problem", namely a Fermat problem where the unknown point is constrained to belong to a line. The resolution of the constrained Fermat problem – by extending the Torricelli method – might suggest a way of solving the Erdős problem, which looks like a non-trivial question.

The Erdős problem can be obviously seen as a so-called fractional programming [5]. However, it is so special that the natural expectation is to be able to construct an "ad hoc" theory. In this sense an open question is the formulation of the "dual" of the Erdős problem, as has happened to the Fermat problem with Fasbender duality [2, 3], which should be contained, as a particular case, in any duality theory for the Erdős problem.

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